

RANK AND BIAS IN FAMILIES OF HYPERELLIPTIC CURVES VIA NAGAO'S CONJECTURE

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Abstract of Report Talk: Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a nontrivial one-parameter family of elliptic curves over \mathbb{Q} with $A(T), B(T) \in \mathbb{Z}(T)$, E_t the specialization of \mathcal{E} to an integer t , $a_t(p)$ be p minus the number of integer solutions to E_t modulo p , and $A_{r,\mathcal{E}}(p) = \sum_{t \bmod p} a_t(p)^r$ be its r -th moment. The first moment $A_{1,\mathcal{E}}(p)$ is related to the rank of the elliptic curve through Nagao's conjecture, which states that $\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{p \leq X} -\frac{1}{p} A_{1,\mathcal{E}}(p) \log(p) = \text{rank}(\mathcal{E}(\mathbb{Q}(T)))$. Computing the rank is important because it gives us information towards the Birch and Swinnerton-Dyer conjecture as well as the structure of the group of rational solutions. The second moment $A_{2,\mathcal{E}}(p)$ is related to rates of convergence in low-lying zeros of L -functions to random matrix theory models. P. Michel proved $A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2})$, if the family \mathcal{E} does not have complex multiplication. In his thesis, S. Miller showed there is a bias in the lower order terms of the second moment of many families of elliptic curves, with the largest lower-order term that does not average to zero having a negative average. This result explains some of the observed excess rank in families.

Nagao's conjecture has a generalization to nontrivial one-parameter families of hyperelliptic curves of any genus. This conjecture is an important heuristic for understanding the rank of the Jacobian of higher genus hyperelliptic curves. In the elliptic curve case, N. Elkies has the record for the largest rank: 28. There are no known approaches or records for the rank of the Jacobian of hyperelliptic curves with higher genus. We explore the higher genus case of $\mathcal{X} : y^2 = x^5 + A(T)x^4 + B(T)x^3 + C(T)x^2 + D(T)x + E(T)$, where $A(T), B(T), C(T), D(T), E(T) \in \mathbb{Z}(T)$. Define X_t , $a_t(p)$, and $A_{r,\mathcal{X}}(p)$ analogously as in the elliptic curve case. Then the generalized Nagao's conjecture is $\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{p \leq X} -\frac{1}{p} A_{1,\mathcal{X}}(p) \log(p) = \text{rank}(J_{\mathcal{X}}(\mathbb{Q}(T)))$. We compute the first and second moments for various families of these surfaces. From the first moments we generalize the constructions by S. Arms, A. Lozano-Robledo, and S. Miller to create a hyperelliptic curve over $\mathbb{Q}(T)$ having Jacobian of moderately large rank; by the specialization theorem of J. Silverman this yields hyperelliptic curves over \mathbb{Q} with large rank Jacobian. Furthermore, we prove a similar bias exists in the expansions of the second moment for these curves; the analysis is complicated by the higher degree Legendre sums that emerge in the passage to hyper-elliptic curves. Lastly, we examine the extent to which Nagao's Conjecture holds for these curves.

[Joint work with Steven J Miller]

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