Sato-Tate in Arithmetic Progressions and Lehmer’s Conjecture

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Abstract of Poster Presentation: In 1916, Ramanujan studied the values of his tau function, which govern fluctuations about the main term in approximating the number of ways to represent a prime as a sum of 24 squares. In 1947, Lehmer conjectured that the Ramanujan tau function, \( \tau(n) \), is nonzero for all \( n \). The tau function also appears as the coefficients of the modular discriminant, which is the prototypical example of a cuspidal newform. A modular form \( f(z) \) of level \( N \) on a congruence subgroup \( \Gamma \) is a cusp form if it vanishes at the cusps of \( \Gamma \); this forces the constant coefficient of its Fourier expansion to equal 0. Intuitively, we say that the cusp form \( f(z) \) is a newform if it does not appear at a lower level than \( N \). It follows from Deligne’s proof of the Weil Conjectures that the coefficients of a newform with weight \( k \) are bounded in absolute value by \( 2p^{(k-1)/2} \); hence we can write them as \( a_f(p) = 2p^{(k-1)/2} \cos(\theta_p) \), for some angle \( \theta_p \in [0,\pi] \). The Sato-Tate conjecture, proven by Barnet-Lamb, Geraghty, Harris, and Taylor, states that these angles are equidistributed with respect to the measure \( d\mu_{ST} = \frac{2}{\pi} \sin^2(\theta) \, d\theta \). Let \( f(z) \in S_{k}^{\text{new}}(\Gamma_0(N)) \) be a newform with squarefree level \( N \) without complex multiplication. In 2017, Rouse and Thorner proved an explicit version of the Sato-Tate conjecture under assumptions on the symmetric power \( L \)-functions of \( f \) which include the Generalized Riemann Hypothesis. They use this explicit result to bound the density of nonzero coefficients of the Ramanujan tau function below by \( 1 - 1.54 \times 10^{-13} \). Under the same assumptions, we prove an explicit version of the Sato-Tate conjecture for primes in arithmetic progressions, and use this result to improve the lower bound found by Rouse and Thorner.

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