Sato-Tate in Arithmetic Progressions and Lehmer’s Conjecture

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Abstract of Poster Presentation: In 1916, Ramanujan studied the values of his tau function, which govern fluctuations about the main term in approximating the number of ways to represent a prime as a sum of 24 squares. In 1947, Lehmer conjectured that the Ramanujan tau function, \(\tau(n)\), is nonzero for all \(n\). The tau function also appears as the coefficients of the modular discriminant, which is the prototypical example of a cuspidal newform. A modular form \(f(z)\) of level \(N\) on a congruence subgroup \(\Gamma\) is a cusp form if it vanishes at the cusps of \(\Gamma\); this forces the constant coefficient of its Fourier expansion to equal 0. Intuitively, we say that the cusp form \(f(z)\) is a newform if it does not appear at a lower level than \(N\). It follows from Deligne’s proof of the Weil Conjectures that the coefficients of a newform with weight \(k\) are bounded in absolute value by \(2p^{(k-1)/2}\), hence we can write them as \(a_f(p) = 2p^{(k-1)/2}\cos(\theta_p)\), for some angle \(\theta_p \in [0, \pi]\). The Sato-Tate conjecture, proven by Barnet-Lamb, Geraghty, Harris, and Taylor, states that these angles are equidistributed with respect to the measure \(d\mu_{ST} = \frac{2}{\pi} \sin^2(\theta) \, d\theta\). Let \(f(z) \in S^\text{new}_k(\Gamma_0(N))\) be a newform with squarefree level \(N\) without complex multiplication. In 2017, Rouse and Thorner proved an explicit version of the Sato-Tate conjecture under assumptions on the symmetric power \(L\)-functions of \(f\) which include the Generalized Riemann Hypothesis. They use this explicit result to bound the density of nonzero coefficients of the Ramanujan tau function below by \(1 - 1.54 \times 10^{-13}\). Under the same assumptions, we prove an explicit version of the Sato-Tate conjecture for primes in arithmetic progressions, and use this result to improve the lower bound found by Rouse and Thorner.

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